# Recall from last time:

Given differentiable functions $f$ and $g$,

1) Product Rule

$$(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$$

2) Chain Rule

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

3) Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

# Application of Chain Rule

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{d}{dx}\left((f(x))^{-1}\right)$$

$$= \boxed{-(f(x))^{-2}\,\frac{d}{dx}(f(x))}$$

$$(f(x) \neq 0)$$

$$\frac{1}{f(x)} = h(g(x))$$

$$\boxed{\begin{aligned} h(x) &= \frac{1}{x} = x^{-1} \\ g(x) &= f(x) \end{aligned}}$$

$$h'(x) = -x^{-2}$$

$$\frac{d}{dx}\left(h(g(x))\right)$$

$$= h'(g(x))\,g'(x)$$

$$= -(f(x))^{-2} f'(x)$$

## Example 1:

$f(x) = (7x^5 + 9x + 1)^{32}$

Find $f'(x)$.

$$f'(x) = 32(7x^5+9x+1)^{31} \cdot (7x^5+9x+1)'$$

$$= \boxed{32(7x^5+9x+1)^{31}(35x^4+9)}$$

$(f(x) = h(g(x)))$

$g(x) = 7x^5 + 9x + 1$

$h(x) = x^{32}$

$g'(x) = 35x^4 + 9$

$h'(x) = 32x^{31}$

By chain rule, $f'(x) = h'(g(x))\, g'(x)$

## Example 2: (How to avoid the quotient rule)

$$f(x) = \frac{\sqrt{4x^4+7x+9}}{11x^2-20x+2}$$

$$f'(x) = ?$$

$$f(x) = \underbrace{\sqrt{4x^4+7x+9}}_{g(x)} \cdot \underbrace{(11x^2-20x+2)^{-1}}_{h(x)}$$

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

(product rule)

$$f(x) = \underbrace{\sqrt{4x^4+7x+9}}_{g(x)} \cdot \underbrace{(11x^2-20x+2)^{-1}}_{h(x)}$$

$$g'(x) =$$

$$h'(x) =$$

$$g(x) = (4x^4+7x+9)^{1/2}$$

$$g'(x) = \tfrac{1}{2}(4x^4+7x+9)^{-1/2} \cdot \underbrace{(16x^3+7)}_{\left(\frac{d}{dx}(4x^4+7x+9)\right)}$$

$$h(x) = (11x^2-20x-2)^{-1}$$

$$h'(x) = -(11x^2-20x-2)^{-2}\,\underbrace{(22x-20)}_{\left(\frac{d}{dx}(11x^2-20x-2)\right)}$$

$$f'(x) = g(x)\,h'(x) + h(x)\,g'(x)$$

$$= \underbrace{(4x^4+7x+9)^{\frac{1}{2}}}_{g(x)} \underbrace{(-1)(11x^2-20x-2)^{-2}(22x-20)}_{h'(x)}$$

$$+ \underbrace{(11x^2-20x-2)^{-1}}_{h(x)} \underbrace{\left(\frac{1}{2}\right)(4x^4+7x+9)^{-\frac{1}{2}}(16x^3+7)}_{g'(x)}$$

# Trigonometric Derivatives and Limits

(Section 2.4)

$f(x) = \sin(x)$.

Find $f'(x)$!

$f'(x) = \cos(x)$, but why?

By definition,

$$\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

trig identity: $\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$

Substitute into sine derivative

$$\frac{d}{dx}(\sin(x))$$

$$= \lim_{h \to 0} \left( \frac{\sin(h)\cos(x)}{h} \right) + \left( \frac{\sin(x)\cos(h) - \sin(x)}{h} \right)$$

$$= \left( \cos(x) \lim_{h \to 0} \frac{\sin h}{h} \right) + \left( \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right)$$

(provided these limits exist!)

Reduce $\frac{\cos(h)-1}{h}$ to $\frac{\sin(h)}{h}$

by multiplying by $\frac{\cos(h)+1}{\cos(h)+1}$

and get $\frac{\cos^2(h)-1}{h(\cos(h)+1)} = \frac{-\sin^2(h)}{h(\cos(h)+1)}$

$$= \frac{\sin(h)}{h} \cdot \frac{-\sin(h)}{\cos(h)+1}$$

? as $h \to 0$

0 as $h \to 0$

## Example 3.

$f(x) = \sin(x^3 - 3)$

Find $f'(x)$.

$f(x) = h(g(x))$

where $g(x) = x^3 - 3$

$h(x) = \sin(x)$

Use chain rule.

$f'(x) = h'(g(x))\, g'(x)$

$= \boxed{\cos(x^3 - 3) \cdot 3x^2}$